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# CLOSED-FORM FORMULAS FOR FUNDAMENTAL VIBRATION FREQUENCY OF BEAMS UNDER OFF-CENTRE LOAD

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## 1. INTRODUCTION

Rayleigh's principle is a common method used to find the natural frequency of vibrating systems. The topic has been well-covered in literature [1–8]. Most examples consider only centrally-loaded beams, except those in references [2], [4] and [5]. Timoshenko *et al.* [2, pp. 38, 39] and Humar [5, p. 296] consider a simply-supported beam of uniform cross-section loaded at position $x = a$, with a block of weight $W$. The static deflection for a light beam carrying a concentrated mass was used in reference [2], whereas the static deflection for a beam with a uniformly distributed load and a concentrated mass was applied to the centrally-loaded beam in reference [5].

Rayleigh's principle states that a reasonable mode shape has to satisfy at least the slope and deflection conditions at the ends, to lead to a good approximation of the natural frequency [1–3]. The accuracy of Rayleigh's method depends on how closely one can estimate the dynamic deflection curve. The static deflection curve is often used to approximate the dynamic deflection for the fundamental mode.

Timoshenko's example [2] uses the strength-of-materials closed solution: $y = Wb/(6lEI)$ $[x^3 - (l^2 - b^2)x]$ to which the sine series converges [3 section 2.7.]. Chai and Low [9] confirmed that a 100-term series is equivalent to the strength-of materials expression. It was found that the expression in reference [2] is applicable only if the load is heavy and is placed near the beam's centre [9]. In fact, if the distributed load is greater than the concentrated mass, it is advisable to assume the static deflection curve for the beam with a uniformly distributed load for the fundamental mode shape [1 section 3.2.].

In this note, different assumed shape functions are used, one at a time, to obtain the kinetic and potential energies of beams carrying a concentrated mass at *various* positions. Three classical beams under off-centre load are considered: simply supported, fixed-fixed, clamped-free (see Figure 1). A closed-form expression for the fundamental frequency of each case is written in terms of mass ratio and position parameter.

## 2. ENERGY METHOD USING DIFFERENT SHAPE FUNCTIONS

By equating $T_{max}$ to $U_{max}$, one obtains Rayleigh's quotient for the fundamental natural frequency of the loaded beams:

$$\omega^2 = EI\int\left(\frac{d^2 y}{dx^2}\right)^2 dx \Big/ \left[\int y^2\, dm + My^2|_{x=a}\right], \tag{1}$$

where $m$ is the mass per unit length of the beam without any load, while $M$ is the mass of the load alone.

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In this work, each of the following shape functions [7, 8] is incorporated into equation (1) to obtain the fundamental frequency: (a) For simply supported conditions,

$$y_{W1} = Wbx/(6lEI)(l^2 - x^2 - b^2), \qquad \text{for } 0 \leqslant x \leqslant a; \tag{2a}$$

$$y_{W2} = Wa(l - x)/(6lEI)(2l - x^2 - a^2), \qquad \text{for } a \leqslant x \leqslant l; \tag{2b}$$

$$y_m = wx/(24EI)(x^3 - 2lx^2 + l^3), \qquad \text{for } 0 \leqslant x \leqslant l; \tag{3}$$

$$y_c = y_m + y_W, \qquad \text{for } 0 \leqslant x \leqslant l; \tag{4}$$

$$y_t = A \sin(\pi x/l), \qquad \text{for } 0 \leqslant x \leqslant l. \tag{5}$$

(b) For fixed-fixed conditons,

$$y_{W1} = Wb^2x^2/(6l^3EI)[3a - x(3a + b)], \qquad \text{for } 0 \leqslant x \leqslant a; \tag{6a}$$

$$y_{W2} = Wa^2(l - x)^2/(6l^3EI)[3bl - (l - x)(3b + a)], \qquad \text{for } a \leqslant x \leqslant l; \tag{6b}$$

$$y_m = wx^2/(24EI)(x^2 - 2lx + l^2), \qquad \text{for } 0 \leqslant x \leqslant l; \tag{7}$$

$$y_c = y_m + y_W, \qquad \text{for } 0 \leqslant x \leqslant l; \tag{8}$$

$$y_t = A[1 - \cos(2\pi x/l)], \quad \text{for } 0 \leqslant x \leqslant l. \tag{9}$$

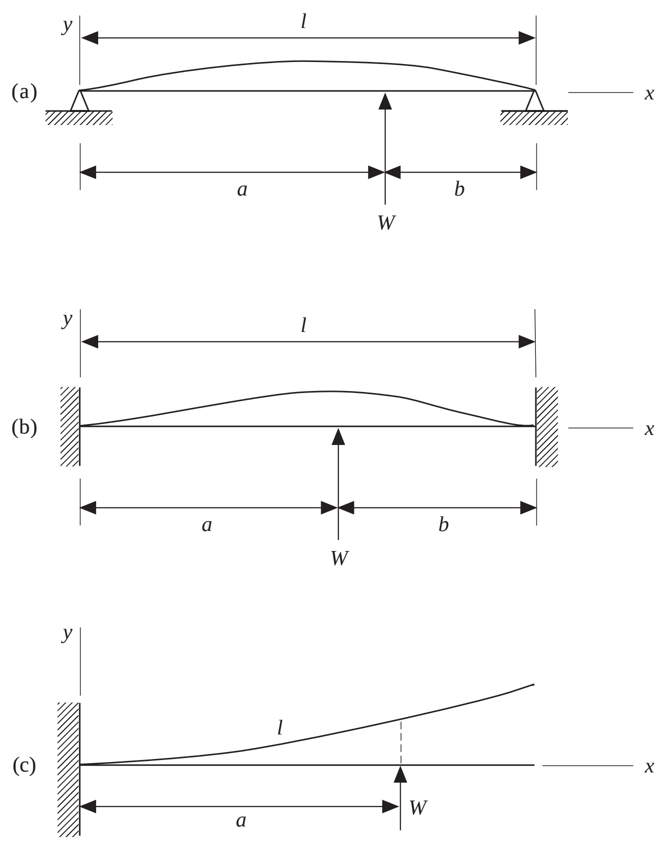


Figure 1. Loaded beam with various ends: (a) simply supported, (b) fixed-fixed, (c) clamped-free.

(c) For clamped-free conditions,

$$y_{W1} = Wx^2/(6EI)(3a - x), \quad \text{for } 0 \leqslant x \leqslant a; \tag{10a}$$

$$y_{W2} = Wa^2/(6EI)(3x - a), \quad \text{for } a \leqslant x \leqslant l; \tag{10b}$$

$$y_m = wx^2/(24EI)(x^2 - 4lx + 6l^2), \quad \text{for } 0 \leqslant x \leqslant l; \tag{11}$$

$$y_c = y_m + y_W, \qquad \text{for } 0 \leqslant x \leqslant l; \tag{12}$$

$$y_t = A[1 - \cos(\pi x/2l)], \qquad \text{for } 0 \leqslant x \leqslant l. \tag{13}$$

Note that $y_{W1}$ and $y_{W2}$ are the deflection curves defined for the left and right portions, respectively. They are static-deflection curves by considering only the load ($W$). On the other hand, the deflection curve $y_m$ is defined in terms of the distributed beam mass ($m$) only, while $y_c$ involves the combined contribution from both the distributed mass and the load. Note that $y_w$ represents the term of $y_{W1}$ or $y_{W2}$ in its respective range. The parameter $A$ of each of trigonometric functions $y_t$ in equations (5), (9), and (13) refers to both the displacement of the beam at $x = a$ and the displacement due to the load.

It is worth mentioning that the deflection curve (for example, $y_W$) may not account for both the beam and the load, even though the terms of $M$ and $m$ are always included in equation (1) for the kinetic energy.

## 3. CLOSED-FORM EXPRESSIONS FOR FUNDAMENTAL FREQUENCY

Now $\omega_W$, $\omega_m$, $\omega_c$ and $\omega_t$ are defined as the associated frequencies obtained from equation (1) by using $y_W$, $y_m$, $y_c$ and $y_t$, respectively. The frequency obtained by using the Rayleigh method can be written in a closed form as

$$\omega^2 = KEI/(Ml^3)(A_\alpha + A)/(B_\alpha + B), \tag{14}$$

in which $K$ is a coefficient to be determined. The parameters $A_\alpha$ and $B_\alpha$ are functions of both the mass ratio ($\alpha = m/M$) and the load's position ($\zeta = x/l$), whereas $A$ and $B$ are independent of $\alpha$.

By virtue of equation (1), the results of these parameters for the respective cases are obtained as follows: (a) for simply supported conditions,

$y_W$: $K = 48; \quad A_\alpha = 0, \; A = 1;$

$$B_\alpha = (16/105)\alpha(3\zeta^4 - 6\zeta^3 - \zeta^2 + 4\zeta + 2), \qquad B = 16\zeta^2(\zeta^2 - 2\zeta + 1).$$

$y_m$: $K = 48; \quad A_\alpha = 0, \; A = 1;$

$$B_\alpha = (31/63)\alpha, \qquad B = 10\zeta^2(\zeta^6 - 4\zeta^5 + 4\zeta^4 + 2\zeta^3 - 4\zeta^2 + 1).$$

$y_c$: $K = 48; \quad A_\alpha = (\alpha/40)(\alpha + 10\zeta^4 - 20\zeta^3 + 10\zeta); \; A = \zeta^2(\zeta^2 - 2\zeta + 1);$

$$\begin{aligned} B_\alpha = \alpha[&(31/2520)\alpha^2 + (9/35)\alpha\zeta^8 - (36/35)\alpha\zeta^7 + \alpha\zeta^6 + (3/5)\alpha\zeta^5 - \alpha\zeta^4 \\ &-(1/5)\alpha\zeta^3 + (1/4)\alpha\zeta^2 + (17/140)\alpha\zeta + (156/35)\zeta^8 - (624/35)\zeta^7 \\ &+(332/15)\zeta^6 - 4\zeta^5 - (136/15)\zeta^4 + 4\zeta^3 + (32/105)\zeta^2], \end{aligned}$$

$$B = 16\zeta^4(\zeta^4 - 4\zeta^3 + 6\zeta^2 - 4\zeta + 1).$$

$y_t$: $K = \pi^4; \quad A_\alpha = 0, \; A = 1; \quad B_\alpha = \alpha, \; B = 2 - 2\cos^2(\pi\zeta).$

(b) for fixed-fixed conditions,

$y_W$: $K = 192;\quad A_\alpha = 0,\ A = 6\zeta^3 - 10\zeta^2 + 2\zeta - 1;\quad B_\alpha = (16/35)\zeta\alpha(\zeta^3 - 2\zeta^2 - 2\zeta + 3),$

$$B = -64\zeta^3(\zeta^3 - 3\zeta^2 + 3\zeta - 1).$$

$y_m$: $$K = 192;\quad A_\alpha = 0,\ A = 1;\ B_\alpha = (21/8)\alpha,$$

$$B = (6615/4)\zeta^4(\zeta^4 - 4\zeta^3 + 6\zeta^2 - 4\zeta + 1).$$

$y_c$: $$K = 192;\ A_\alpha = \alpha[(1/8)\alpha + (15/2)\zeta^4 - 15\zeta^3 + (15/2)\zeta^2],$$

$$A = -30\zeta^3(\zeta^3 - 3\zeta^2 + 3\zeta - 1);$$

$$B_\alpha = (2/7)\alpha[(1/6)\alpha^2 + 108\alpha\zeta^8 - 432\alpha\zeta^7 + 644\alpha\zeta^6 - 420\alpha\zeta^5 + 105\alpha\zeta^4 - 14\alpha\zeta^3 + 9\alpha\zeta^2 - 1728\zeta^{10} + 8640\zeta^9 - 17136\zeta^8 + 16704\zeta^7 - 7776\zeta^6 + 1152\zeta^5 + 144\zeta^4],$$

$$B = 1920\zeta^6(\zeta^6 - 6\zeta^5 + 15\zeta^4 - 20\zeta^3 + 15\zeta^2 - 6\zeta + 1).$$

$y_t$: $K = \pi^4;\ A_\alpha = 0,\ A = 1;\quad B_\alpha = (3/16)\alpha,\ B = (1/8)[\cos^2(2\pi\zeta) - 2\cos(2\pi\zeta) + 1].$

(c) for clamped-free conditions,

$y_W$: $K = 3;\quad A_\alpha = 0,\ A = 1;\ B_\alpha = -\zeta\alpha[(1/70)\zeta^3 - (1/4)\zeta^2 + (3/4)\zeta - (3/4)],\ B = \zeta^3$ .

$y_m$: $$K = 3;\quad A_\alpha = 0,\ A = 1;\quad B_\alpha = (13/54)\alpha,$$

$$B = \zeta^4[(5/48)\zeta^4 - (5/6)\zeta^3 + (35/12)\zeta^2 - 5\zeta + (15/4)].$$

$y_c$: $$K = 3;\quad A_\alpha = (1/4)\alpha[(3/5)\alpha + \zeta^4 - 4\zeta^3 + 6\zeta^2],\ A = \zeta^3;$$

$$B_\alpha = \alpha[(13/360)\alpha^2 + (9/560)\alpha\zeta^8 - (9/70)\alpha\zeta^7 + (9/20)\alpha\zeta^6 - (3/4)\alpha\zeta^5 + (9/16)\alpha\zeta^4 - (3/20)\alpha\zeta^3 + (13/40)\alpha\zeta^2 + (33/140)\zeta^7 - (3/4)\zeta^6 + (3/4)\zeta^5 + (3/4)\zeta^4],\ B = \zeta^6.$$

$y_t$: $$K = \pi^5;\quad A_\alpha = 0,\ A = 1;\quad B_\alpha = 16\alpha(3\pi - 8),$$

$$B = 32\pi[\cos^2(\pi\zeta/2) - 2\cos(\pi\zeta/2) + 1].$$

The expression can be simplified if the load is placed at the particular position (i.e., $\zeta = 0{\cdot}5$ or 1):

(a) for simply supported conditions ($\zeta = 0{\cdot}5$),

$y_W$: $$K = 48;\quad A_\alpha = 0,\ A = 1;\quad B_\alpha = (17/35)\alpha,\ B = 1.$$

$y_m$: $$K = 48;\quad A_\alpha = 0,\ A = 1;\quad B_\alpha = (31/63)\alpha,\ B = 125/128.$$

$y_c$: $$K = 48;\quad A_\alpha = (2/5)\alpha^2 + (5/4)\alpha,\ A = 1;$$

$$B_\alpha = \alpha[(62/315)\alpha^2 + (113/112)\alpha + (243/140)],\ B = 1.$$

$y_t$: $$K = \pi^4;\quad A_\alpha = 0,\ A = 1;\quad B_\alpha = \alpha,\ B = 2.$$

(b) for fixed-fixed conditions ($\zeta = 0{\cdot}5$),

$y_W$: $$K = 192;\quad A_\alpha = 0,\ A = 1;\quad B_\alpha = (13/35)\alpha,\ B = 1.$$

$y_m$: $$K = 192;\quad A_\alpha = 0,\ A = 1;\quad B_\alpha = (8/21)\alpha,\ B = 15/16.$$

$y_c$: $K = 192$; $\quad A_\alpha = (4/15)\alpha^2 + \alpha,\ A = 1$; $\quad B_\alpha = \alpha[(32/315)\alpha^2 + (67/105)\alpha + (48/35)]$,

$B = 1$.

$y_t$: $\quad K = \pi^4$; $\quad A_\alpha = 0,\ A = 1$; $\quad B_\alpha = (3/16)\alpha,\ B = 1/2$.

(c) for clamped-free conditions ($\zeta = 1$),

$$y_W\text{: } K = 3; \quad A_\alpha = 0,\ A = 1; \quad B_\alpha = (33/140)\alpha,\ B = 1.$$

$$y_m\text{: } K = 3; \quad A_\alpha = 0,\ A = 1; \quad B_\alpha = (13/54)\alpha,\ B = 15/16.$$

$$y_c\text{: } K = 3; \quad A_\alpha = (3/20)\alpha^2 + (3/4)\alpha,\ A = 1;\ B_\alpha = \alpha[(13/360)\alpha^2 + (13/40)\alpha + (69/70)],$$

$$B = 1.$$

$$y_t\text{: } K = \pi^5; \quad A_\alpha = 0,\ A = 1;\ B_\alpha = 16\alpha(3\pi - 8),\ B = 32\pi.$$

The result for the end-loaded fixed-fixed beam,

$$\omega_W^2 = 3EI/(Ml^3)/[(33/140)\alpha + 1],$$

agrees with the familiar form found in the literature. Another result for the case of the centrally loaded simply supported beam is identical to the expression by considering both the distributed mass and the load in reference [5, p. 296],

$$\omega_c^2 = \frac{48EI}{Ml^3}\frac{(2/5)\alpha^2 + (5/4)\alpha + 1}{(62/315)\alpha^3 + (113/112)\alpha^2 + (243/140)\alpha + 1}.$$

In fact, the corresponding equivalent mass and stiffness, by using $y_W$, have been summarized in references [2, 7, 8] for different loaded beams:

simply supported: $\zeta = 0{\cdot}5$, $\quad m_{eq} = M + (17/35)m$, $\quad k_{eq} = 48EI/l^3$;

fixed-fixed: $\zeta = 0{\cdot}5$, $\quad m_{eq} = M + (13/35)m$, $\quad k_{eq} = 192EI/l^3$;

clamped-free: $\zeta = 1$, $\quad m_{eq} = M + (33/140)m$, $\quad k_{eq} = 3EI/l^3$.

Several points are worth noting when comparing the frequency expressions given here. First the complexity of the expressions increases significantly if the load is placed away from the centre, mainly due to the presence of a position function, $\zeta$. Second, it is obvious that the linear summation of $y_c(=y_m + y_W)$ does not lead to a summation for $\omega_c$. In fact, the expression is much more complicated if both the beam and load are considered by using $y_c$.

## 4. CONCLUDING REMARKS

Closed-form expressions for the fundamental frequency of beams carrying a mass at various positions have been obtained by using Rayleigh's quotient. Beams with end conditions of simply supported, fixed-fixed and clamped-free are considered. For the shape functions, a trigonometric function and the three deflection curves involving the mass/load are used. Although the function $y_W$ is commonly used, it is recommended to use $y_c$ for every case unless the beam's mass is negligible. By virtue of the closed-form expression, one can quickly evaluate the fundamental frequency of the off-centre loaded beams by substituting the corresponding mass ratio and position parameter into equation (14).

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REFERENCES

1. G. B. WARBURTON 1964 *The Dynamical Behaviour of Structures*. Oxford: Pergamon Press.
2. S. P. TIMOSHENKO, D. H. YOUNG and W. WEAVER, JR. 1974 *Vibration Problems in Engineering*, Section 1.4. New York: John Wiley; fourth edition.
3. L. H. DONNELL 1976 *Beams*, *Plates and Shells*. New York: McGraw–Hill.
4. R. D. BLEVINS 1979 *Formulas for Natural Frequency and Mode Shape*. New York: Van Nostrand Reinhold.
5. J. L. HUMAR 1990 *Dynamics of Structures*. Section 8-5. Englewood Cliffs, NJ: Prentice Hall.
6. WILLIAM T. THOMSON 1993 *Theory of Vibration with Applications*, example 10.8-2. Englewood Cliffs, NJ: Prentice Hall; fourth edition.
7. M. L. JAMES, G. M. SMITH, J. C. WOLFORD and P. W. WHALEY 1994 *Vibration of Mechanical and Structural Systems*: *with Microcomputer Applications*. New York: Harper Collins.
8. SINGIRESU S. RAO 1995 *Mechanical Vibrations*. Reading, MA: Addison–Wesley; third edition.
9. G. B. CHAI and K. H. LOW 1993 *Journal of Sound and Vibration* **160**, 161–166. On the natural frequencies of beams carrying a concentrated mass.